- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges $\Leftrightarrow p>1$
- $\sum_{n=1}^{\infty} a_n$ Converges $\Rightarrow \lim_{n\to\infty} a_n = 0$
- $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q|<1$
- $\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}$
- $0 \leq a_n \leq b_n$ ① $\sum b_n$ converges $\Rightarrow \sum a_n$ converges
  ② $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

Field Axioms

$\forall x,y \in \mathbb{R}$ $x+y=y+x$, $x \cdot y = y \cdot x$ (Commutativity) $\exists 0 \neq 1 \in \mathbb{R}$ s.t. $x+0=x$, $x \cdot 1 = x$ $\forall x \in \mathbb{R}$ (identity)

$\forall x,y,z \in \mathbb{R}$ $x+(y+z)=(x+y)+z$ (associativity) $\forall x\, \exists(-x)$ s.t. $x+(-x)=0$ (negatives)

$\forall x,y,z \in \mathbb{R}$ $x\cdot(y+z) = x\cdot y + x \cdot z$ (distributivity) $\forall x \neq 0\, \exists x^{-1}$ s.t. $x(x^{-1})=1$ (inverses)

* $0, 1, -x, x^{-1}$ unique.

* $(-x)\cdot y = -(x \cdot y)$, $-(-x)=x$, $(x^{-1})^{-1}=x$, $x\cdot y = 0 \Rightarrow x=0 \vee y=0$, $(xy)^{-1}=x^{-1}\cdot y^{-1}$ $(x,y\neq 0)$, $-0=0$, $1^{-1}=1$,

$(x+y)^2 = x^2+2xy+y^2$, $-(x+y)=(-x)+(-y)$

Order Axiom $\mathbb{R}^+ \subseteq \mathbb{R}$; (1) $\forall x,y \in \mathbb{R}^+$, $x+y \in \mathbb{R}^+$, $x\cdot y \in \mathbb{R}^+$ (Closure)

* $\forall \varepsilon>0$, $a \geq b-\varepsilon \Rightarrow a \geq b$ (2) $\forall x \in \mathbb{R}$, exactly one of $x \in \mathbb{R}^+$, $-x \in \mathbb{R}^+$, $x=0$ (trichotomy)

* $x \geq y \Rightarrow x-y \in \mathbb{R}^+ \cup \{0\}$. $\forall x\; x \leq x$ (Reflexive); $\forall x,y\; x\leq y \wedge y \leq x \Rightarrow x=y$ (antisymmetry); $\forall x,y,z\; x\leq y \wedge y \leq z \Rightarrow x \leq z$ (Transitivity).

* $a>b \wedge c\geq d \Rightarrow a+c>b+d$, $(a>b>0)\wedge(c\geq d>0) \Rightarrow a\cdot c > b \cdot d$, $(a>b)\wedge(c>0) \Rightarrow a\cdot c < b \cdot c$, $0<a<1 \Rightarrow \frac{1}{a}>1$, $a<b \Rightarrow a<\frac{a+b}{2}<b$

* $|a| \geq 0$, $|a| = \max\{a,-a\}$, $|a\cdot b| = |a|\cdot|b|$, $|a+b| \leq |a|+|b|$, $||a|-|b|| \leq |a-b|$, $(r>0)\wedge(|a-b|<r) \Leftrightarrow b \in (a-r, a+r)$, $\forall \varepsilon>0\, \exists n \in \mathbb{N}\; \frac{1}{n}<\varepsilon$

Least Upper Bound Axiom $K \subseteq \mathbb{R}$, $x \in \mathbb{R}$ upper bound $\Leftrightarrow x \geq k\; \forall k \in K$. every nonempty set of $\mathbb{R}$ bounded above has a Supremum.

$-\sup(K)=b \Leftrightarrow b$ upper bound and $b \leq u$ $\forall u$, upper bound for $K$. , $b \in K \Rightarrow \max(K)=b$.

$b=\sup(K) \Leftrightarrow \forall \varepsilon>0\, \exists x \in K$ s.t. $|x-b|<\varepsilon \Leftrightarrow \forall \varepsilon>0\, \exists x \in K$ s.t. $x \in (b-\varepsilon, b]$

$b=\inf(K) \Leftrightarrow \forall \varepsilon>0\, \exists x \in K$ s.t. $|x-b|<\varepsilon \Leftrightarrow \forall \varepsilon>0\, \exists x \in K$ s.t. $x \in [b, b+\varepsilon)$

(Archemedian Property) $\forall x \in \mathbb{R}\, \exists n \in \mathbb{N}\; n>x$ $\quad x,y \in \mathbb{R}$ $x \leq y+\varepsilon\; \forall \varepsilon>0 \Rightarrow x \leq y$

$[a_1,b_1] \supseteq [a_2,b_2] \supseteq \ldots$ $(a_i \leq b_i\; \forall i \in \mathbb{N}) \Rightarrow \bigcap_{i=1}^{\infty}[a_i,b_i] \neq \emptyset$ (Nested Interval theorem)

Metric 1) $D(x,y)\geq 0$ 2) $D(x,y)=0 \Leftrightarrow x=y$ 3) $D(x,y)=D(y,x)$ 4) $D(x,z) \leq D(x,y)+D(y,z)$

Open ball $B_r(a)=\{x \in X : d(a,x)<r\}$, $C_r(a)=\{x\in X: d(a,x)\leq r\}$

Open Set: $(X,d)$ metric space, $U=\bigcup_{\alpha \in I} B_{r_\alpha}(a_\alpha)$, $a_\alpha \in X$, $r_\alpha>0$ $\forall \alpha \in I$. Arbitrary unions and finite intesections of open sets are open.

$U$ open set in $X \Leftrightarrow \forall a \in U\, \exists r>0 : B_r(a) \subseteq U$

$\operatorname{diam}(S) = \{0$ if $S=\emptyset$, $\sup\{d(a,b): a,b \in S\}$ if $S$ is bounded above, $\infty$ if $S$ not bounded above.$\}$

$\operatorname{diam}(S) \neq \infty \Leftrightarrow S$ bounded $\Leftrightarrow S \subseteq B_r(a)$, some $a \in X$, $r>0 \Leftrightarrow \forall a \in X\, \exists r>0 : S \subseteq B_r(a)$.

Convergence of Sequences – $\lim_{n\to\infty} a_n = x$ if $\forall \varepsilon>0\, \exists N \in \mathbb{N}\, \forall n>N\; d(a_n,x)<\varepsilon$, and $\lim_{n\to\infty} a_n \neq x$ if $\exists \varepsilon>0\, \forall N \in \mathbb{N}\, \exists n>N\; d(a_n,x)\geq\varepsilon$

– $a_N, a_{N+1}, \ldots$ Nth tail of $(a_n)$. If tail of $a_n \to x$ then $a_n \to x$. $a_n$ eventually Constant if it has a tail that is Constant. $(a_n=c\; \forall n>N)$. $\lim_{n\to\infty} a_n = x \Leftrightarrow$ for all open sets $U$ with $x \in U$ $\exists N \in \mathbb{N}\, \forall n>N$, $a_n \in U \Leftrightarrow \forall \varepsilon>0\, \exists N \in \mathbb{N}\, \forall n>N\; a_n \in B_\varepsilon(x)$

limits are unique. Convergent Sequences are bounded. $\emptyset \neq S \subseteq \mathbb{R}$ if $\sup(S)=l$, there is a Sequence Converging to it (likewise for $\inf(S)$)

If $a_n \to L$, $b_n \to M$, then $(a_n \pm b_n) \to L \pm M$, $(a_n \cdot b_n) \to L \cdot M$, $\left(\frac{a_n}{b_n}\right) \to \frac{L}{M}$ $(b_n)\neq 0$, $M \neq 0$.

Limit Point – Given $(X,d)$, $S \subseteq X$ $x \in X$ is limit point if there is a Sequence in $S\setminus\{x\}$ converging to $x$. $\Leftrightarrow \forall r>0$, $B_r(x)$ Contains a point from $S \setminus \{x\}$

Closed Set – $C \subseteq X$ closed if it contains all of its limit points $\Leftrightarrow a_n$ in $C$ and $a_n \to x$, then $x \in C$. $\Leftrightarrow X \setminus C$ open in $X$.

$\overline{C}$ – Intersection of all closed Subsets in $X$ Containing $C$. $\operatorname{int}(C)$ – Set of all interior points of $C$ – $x \in C$ Such that $\exists r>0 : B_r(x) \subseteq C$. $\operatorname{int}(C) \subseteq C \subseteq \overline{C}$, $C=\overline{C}$ if $C$ closed, $\operatorname{int}(C)=C$ if $C$ open.

$x \in \overline{C} \Leftrightarrow \forall r>0\; B_r(x) \cap C \neq \emptyset$. $\overline{C}=C \cup C'$ ($C'$ = Set of limit points of $C$).

Limits of functions: $(X, d_X), (Y, d_Y)$ $f: X \to Y$, $\lim_{x\to a} f(x) = L$ if $\forall \varepsilon > 0\ \exists \delta > 0: 0 < d_X(x,a) < \delta \Rightarrow d_Y(f(x), L) < \varepsilon$

$\lim_{x\to a} f(x) = L \Leftrightarrow$ for every $(x_n)$ in $X \setminus \{a\}$, if $x_n \to a$, $f(x_n) \to L$.

Continuity: f Continuous at $a$ if $\lim_{x\to a} f(x) = f(a) \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0\ d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon \Leftrightarrow$ for any $(x_n)$, $x_n \to a \Rightarrow f(x_n) \to f(a)$.

$f: X \to Y$ continuous $\Leftrightarrow \forall V \subseteq Y$ if $V$ open then $f^{-1}(V) = \{x \in X: f(x) \in V\}$ is open in $X$.

$f: K \to \mathbb{R}$ $K \subseteq X$, if $f(x) \geq t$ $\forall x \in B_r^*(a) \cap K$ ($a$ = limit point of $K$, $t \in \mathbb{R}$), then $\lim_{x\to a} f(x) \geq t$ as long as it exists.

if $g: K \to \mathbb{R}$ and $f(x) \geq g(x)$ $\forall x \in B_r^*(a) \cap K \Rightarrow \lim_{x\to a} f(x) \geq \lim_{x\to a} g(x)$ if limits exist.

if $h: K \to \mathbb{R}$ and $f(x) \leq g(x) \leq h(x)$ $\forall x \in B_r^*(a) \cap K \Rightarrow (\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} g(x) = L)$

Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} [f(x) \pm g(x)] = L \pm M$, $\lim_{x\to a} [f(x) \cdot g(x)] = L \cdot M$, if $M \neq 0$, $g(x) \neq 0$ on $B_r^*(a) \cap K$ for for some $r > 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $|\lim_{x\to a} f(x)| = \lim_{x\to a} |f(x)|$

Cauchy Sequences $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m > N\ d(a_n, a_m) < \varepsilon$. $(a_n)$ Convergent $\Rightarrow$ $(a_n)$ Cauchy. [Converse true in $\mathbb{R}^n$]

$(a_n)$ Cauchy $\Rightarrow$ $(a_n)$ Convergent Subsequence, $(a_n)$ bounded Sequence in $\mathbb{R}^n \Rightarrow (a_n)$ Convergent Subsequence. [Bolzano-Weierstrauss]

Complete Metric Space - $(X, d)$ Complete if every Cauchy Sequence in $X$ Converges to a point in $X$.

Compact Sets - Every open Cover has a finite subcover. - Finite sets, $[a,b]$ (in $\mathbb{R}$) are compact

$X$ metric space, $C \subseteq X$ then $C$ compact $\Rightarrow$ $C$ closed and bounded [Converse true in $\mathbb{R}^n$] [Heine-Borel]

$C$ compact, $A \subseteq C$ closed $\Rightarrow A$ compact. $C$ compact $\Leftrightarrow$ Every Sequence in $C$ has a subsequence that converges to a point in $C$ [Sequential compactness].

$X, Y$ metric spaces and $S \subseteq X$ compact, $f$ continuous $\Rightarrow f(S) \subseteq Y$ compact.

$X \neq \phi$ Compact metric space, $f: X \to \mathbb{R}$ Continuous $\Rightarrow f$ has an absolute max and absolute min [Extreme Value Theorem]

Uniform Continuity - $f: X \to Y$, $S \subseteq X$, $\forall \varepsilon > 0\ \exists \delta\ \forall x_1, x_2 \in S: d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$, [$f$ uniformly Continuous on $S$]

uniform Continuity $\Rightarrow$ Continuity

Extreme Value Theorem - Let $X = \phi$ Compact metric Space, $f: X \to \mathbb{R}$ a continuous function. $f$ has an absolute max and min.

Let $f: X \to Y$ continuous and $X$ compact $\Rightarrow$ $f$ uniformly Continuous on $X$.

Riemann Integrable - $f$ riemann integrable on $[a,b]$, provided there is an $I \in \mathbb{R}$, s.t. $\forall \varepsilon > 0\ \exists \delta > 0\ \forall P: \|P\| < \delta$ and for any set of Sampling points $\{c_i\}_{i=0}^{n}$ $|R(f,P) - I| < \varepsilon$; where $P = \{x_0, \dots, x_n\}$, $a = x_0 < x_1 < \dots < x_n = b$, $\|P\| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}$, $c_i \in [x_{i-1}, x_i]$, $R(f,P) = \sum_{i=1}^{n} f(c_i) \cdot (x_i - x_{i-1})$; $I = \int_a^b f(x)dx \leftarrow$ Reimann Integral of $f$.

$f: [a,b] \to \mathbb{R}$ integrable $\Rightarrow$ integral is unique, $f$ is bounded function.

$f: [a,b] \to \mathbb{R}$ integrable $\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0: \|P\| < \delta\ \forall R_1(f,P), R_2(f,P)\ |R_1(f,P) - R_2(f,P)| < \varepsilon$.

$f: [a,b] \to \mathbb{R}$ Continuous $\Rightarrow f$ integrable, $f: [a,b] \to \mathbb{R}$ monotonic $\Rightarrow f$ integrable.

$f: [a,b] \to \mathbb{R}$ integrable, $R(f, P_n)$ Sequence of Riemann Sums: $\|P_n\| \xrightarrow[n\to\infty]{} 0 \Rightarrow (R(f, P_n)) \to \int_a^b f(x)dx$.

$f, g: [a,b] \to \mathbb{R}$ integrable $\Rightarrow \int_a^b f(x) \pm g(x) dx = \int_a^b f(x)dx + \int_a^b g(x)dx$, $f \cdot g$ integrable

$\int_a^b k f(x) dx = k \int_a^b f(x) dx$

$f(x) \geq g(x) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

$|f|$ integrable and $\int_a^b |f(x)| dx \geq |\int_a^b f(x) dx|$

**Topologies** $(M, \mathcal{J})$ $M$ set, $\mathcal{J} \subseteq P(M)$ $X \in \mathcal{J} \Leftrightarrow X$ open ① $M \in \mathcal{J}, \emptyset \in \mathcal{J}$ ② $V_1, V_2 \in \mathcal{J} \Rightarrow V_1 \cap V_2 \in \mathcal{J}$ ③ $V_i \in \mathcal{J} \Rightarrow \bigcup_I V_i \in \mathcal{J}$ (any $I$)

U1 → using $\{V \mid \forall p \in V, \exists r > 0, B_r(p) \subseteq V\}$ as a topology on M.

**Metric spaces** $(M, d)$ $d: M \times M \to [0, \infty)$ ① $d(x,y) = d(y,x)$ ② $d(x,z) \leq d(x,y) + d(y,z)$ ③ $d(x,x) = 0$, $d(x,y) > 0 \Rightarrow x \neq y$ $\forall x, y, z \in M$.

**Continuity**
- → topological $\forall V \subseteq M$ open $\Rightarrow f^{-1}(V) \subseteq M$ open, ↔ for every closed $C \subseteq N$, $f^{-1}(C)$ closed in M. ↔ $p \in M, V \subseteq N$ open, $f(p) \in V$, $\exists U \subseteq M$ open s.t. $p \in U$ and $f(U) \subseteq V$.
- → sequential $(p_n) \to p$ in $M \Rightarrow (f(p_n)) \to f(p)$ $f: M \to N$
- → metric $\forall p \in M\ \forall \varepsilon > 0\ \exists \delta > 0\ d_M(p,q) < \delta \Rightarrow d_N(f(p), f(q)) < \varepsilon$

* topological ⇒ sequential always
* all equivalent in metric spaces

$\boxed{\text{Uniform Cont.}}$ f cont., M Compact ⇒ f uniform continuous.

**Closed sets**
- → topological $A \subseteq M$, $M \setminus A$ open
- → Sequential A contains all its limit points

* closed ⇒ seq. closed always
closed = seq. closed for metric spaces

**Convergence**
$\forall \varepsilon > 0\ \exists n_0, n > n_0 \Rightarrow d(p_n, p) < \varepsilon \Leftrightarrow p_n \to p$
⇓
$\forall \varepsilon > 0\ \exists n_0, n, m > n_0 \Rightarrow d(p_n, p_m) < \varepsilon \Leftrightarrow$ Cauchy

Cauchy ⇒ Convergent (in $\boxed{\text{Complete space}}$) → Closed subspace of Complete space is complete. → Union of complete spaces is complete.
⇒ bounded, has convergent subsequence.

**Compactness**
- → sequential – Every seq. has a convergent subsequence.
- → covering – Every open cover has finite subcover ↕ in metric spaces only
  - open cover: $\mathcal{C} \subseteq P(M)$, $A \subseteq \bigcup_{V \in \mathcal{C}} V$, $\forall V$ open in M
  - subcover: $\mathcal{C}' \subseteq \mathcal{C}$

A seq. compact ⇒ A seq. closed
A bounded
f seq. cont ⇒ f(A) seq. compact, f has max and min.
M seq. compact, A seq. closed ⇒ A seq. compact
A complete

**Homeomorphism** $f: M \to N$ s.t. ① f bijection ② f cont. ③ $f^{-1}$ cont. ; f continuous, bijection, M seq. compact ⇒ f homeomorphism.

**Embedding** $f: A \to N$, s.t. $f: A \hookrightarrow f(A)$ is a homeomorphism.
equivalently, $f^{-1}$ cont $\Longleftrightarrow \forall V \subseteq M$ open $\Rightarrow f(V)$ open $\Longleftrightarrow \forall K \subseteq M$ closed $\Rightarrow f(V)$ closed

$\boxed{\text{Intrinsic}}$ A is P ⇔ ($\forall$ embedding $f: A \to N$, f(A) is P)

**Product Topology** $N \times N \to [0, \infty)$ $d_N((x,y),(x',y')) = d(x,y) + d'(x',y') \leftrightarrow \sqrt{d(x,y)^2 + d'(x',y')^2} \leftrightarrow \max\{d(x,y), d'(x',y')\}$ all induce same topology!
$N = M \times M'$, $(M, d)$, $(M', d')$

**Bolzano-Weierstrass** M metric space $\forall p \in M\ \forall r > 0\ \overline{B_r(p)}$ seq. compact ⇒ Bolzano-weierstrass: $(p_n)$ bounded ⇒ $(p_n)$ has convergent subsequence ⇒ $\boxed{\text{Heine-Borel}}$ $A \subseteq M$ seq. compact ⇔ A closed and bounded.

**Connected**
- → $A \amalg B = M$, $U \subseteq M$ connected ⇒ $U \subseteq A$ or $U \subseteq B$, $I \subseteq \mathbb{R}$ connected ⇔ I interval → $\forall a < b < c$, $a, c \in I \Rightarrow b \in I$
- → Intrinsic property of M
- → M has a proper clopen subset ⇔ M has a disconnection → $A \amalg B = M$, $A, B \neq \emptyset$. ($A, B \subseteq M$ clopen)

$\boxed{\text{IVT}}$ $f: M \to N$ continuous, M connected ⇒ f(M) connected

**Subspace** → $A \subseteq M$ has following topology $\{V : V = W \cap A, W \text{ open in } M\}$ * $S \subseteq T \subseteq \overline{S}$, S connected ⇒ T connected.

**Path Connected** → $\forall a, b \in M\ \exists$ path in M (path → continuous mapping $\gamma: [0,1] \to M$ s.t. $\gamma(0) = a$, $\gamma(1) = b$), path connected ⇒ connected, $A \subseteq \mathbb{R}^n$, A open, connected ⇔ A path connected.

**Dense set** A dense in M ⇔ $\forall x \in M\ \forall \varepsilon > 0\ \exists a \in A: d(x,a) < \varepsilon$. ⇔ $\overline{A} = M$

**Function Space** $C_b(A) = \{f: A \to \mathbb{R} \mid f \text{ bounded}\}$, $(C_b(A), d_\infty)$ is a metric space, where $\to d_\infty(f,g) = \sup_{x \in A} |f(x) - g(x)| = \|f-g\|_\infty$

$\hookrightarrow$ Is complete. $\to \|f\|_\infty = \sup_{x\in A} \{|f(x)|\}$

$\hookrightarrow C_0(A) = \{f: A \to \mathbb{R} \mid f \text{ continuous, bounded}\}$

$\hookrightarrow C_0(A)$ closed in $C_b(A) \Rightarrow C_0(A)$ complete.

$\hookrightarrow R = \{$Set of Reimann Integrable functions$\}$ closed in $C_b(A)$, $f: R \to \mathbb{R}$ s.t. $f \mapsto \int_a^b f$ continuous function.

**$(f_n)$ Convergence**

$f_n \to f \quad \forall x \in A\ \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N} \quad n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$

$f_n \rightrightarrows f \Leftarrow \begin{cases} f_n \to f,\ f_n \in C_b(A) \text{ Complete.} \\ f_n \text{ Uniformly Cauchy} \end{cases}$

$f_n \rightrightarrows f \quad \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N} \quad n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$

$f_n \rightrightarrows f \Rightarrow f_n$ uniformly Cauchy

$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon$

$f_n \rightrightarrows f$, $f_n$ cont. $\forall n \Rightarrow f$ cont.

**$\sum_{k=1}^{n} f_k$ Convergence**

$F_n(x) = \sum_{k=0}^{n} f_n(x)$ sequence of partial sums, $\sum_{k=0}^{\infty} f_k(x) := \lim_{n\to\infty} F_n(x) = F(x)$

$\sum_{k=0}^{\infty} f_k(x)$ converges pointwise / uniformly / absolutely / uniformly-absolutely when $F_n(x)$ converges pointwise / uniformly / absolutely ($\sum |f_k(x)|$ converges pointwise) / uniformly-absolutely ($\sum |f_k(x)|$ converges uniformly)

uniform convergence + absolute convergence $\not\Rightarrow$ uniformly-absolutely [e.g. $f_n(x) = x^n/n$ on $(-1,0)$]

uniformly absolutely convergent in $C_0 \Rightarrow$ uniform convergent

Weierstrass M-test: $\sum M_k$ convergent series of constants, $f_k \in C_b$ has $\|f_k\| \leq M_k \Rightarrow \sum f_k$ converges uniformly-absolutely.

**$(f_n)$ Cauchy**

Uniform Cauchy: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall x \in A\ n, m > N \Rightarrow d(f_n(x), f_m(x)) < \varepsilon$

Pointwise Cauchy: $\forall x \in A\ \forall \varepsilon > 0\ \exists n \in \mathbb{N}$ s.t. $n, m > N \Rightarrow d(f_n(x), f_m(x)) < \varepsilon$

Complete space and uniform Cauchy $\Rightarrow$ uniform convergent

Complete space and pointwise Cauchy $\Rightarrow$ pointwise convergent

$f_n \to f$ and $f_n$ uniform Cauchy $\Rightarrow f_n \rightrightarrows f$

**Riemann Integration**

$f$ continuous on compact domain $\Downarrow$

$f$ Reimann Integrable $\Leftrightarrow \forall \varepsilon > 0\ \exists P$ partition s.t. $U(f,P) - L(f,P) < \varepsilon$, where $U(f,P) := \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in (x_i - x_{i-1})} f(x)$

and $L(f,P) \leq U(f,P) \leq (b-a)\|f\|_\infty$ $\qquad L(f,P) := \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in (x_i - x_{i-1})} f(x)$

$f_n \rightrightarrows f$, $f_n$ Reimann integrable $\Rightarrow$ $f$ Riemann integrable, $\int f_n(x)\,dx \to \int f(x)\,dx$

**Differentiation**

$\phi(t) = \frac{f(t) - f(x)}{t - x}$, $t \neq x$; $g(t)$, $t = x$

$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$, $t \neq x$; $f_n'(t)$, $t = x$

$f$ differentiable at $x \Leftrightarrow \phi(t)$ continuous at $x \Leftarrow \begin{cases} \phi_n(t) \text{ cont. } \forall n \\ \phi_n(t) \rightrightarrows \phi(t) \end{cases} \Leftarrow \begin{cases} \phi_n \to \phi \\ \phi_n \text{ Cauchy} \end{cases}$

[Can show $f_n'$ uniformly Cauchy]

$A$ open in $\mathbb{R}$, $f_n \to f$, $f_n$ differentiable $\forall n$, $f_n' \rightrightarrows g \Rightarrow f$ differentiable, $f' = g$.

MVT: $f$ continuous on $[a,b]$, differentiable on $(a,b)$, $\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b-a}$

**Power Series**

$f(x) = \sum c_k x^k$, $c_k \in \mathbb{R}\ \forall k$. $\to \exists R \geq 0$ s.t. $f(x)$ converges absolutely $\forall x \in (-R, R)$.

$f(x)$ diverges if $|x| > R$.

$\forall r < R$, $f(x)$ converges uniformly on $[-r, r]$

$f'(x) = \sum c_k k x^{k-1}$, $\int f(x)\,dx = \sum c_k \frac{1}{k+1} x^{k+1}$

**The Space $C_0([a,b])$**

$\forall f \in C_0$, $f$ bounded function ($f$ cont., $[a,b]$ compact $\Rightarrow f([a,b])$ compact $\Rightarrow f$ bounded)

let $\mathcal{E} \subseteq C_0$, $\mathcal{E}$ pointwise bounded if $\exists \phi(x) < \infty$ s.t. $\forall f \in \mathcal{E}\ |f(x)| < \phi(x)$

$\mathcal{E}$ uniformly bounded if $\exists M > 0$ s.t. $\forall f \in \mathcal{E}$, $\|f\| < M$

$\to$ Pointwise + Compact domain $\Rightarrow$ uniform equicontinuous

$\mathcal{E}$ pointwise equicontinuous if $\forall s \in [a,b]\ \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall t \in [a,b]\ \forall f \in \mathcal{E}\ |s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$

$\Rightarrow$ lipschitz if $|f(x) - f(y)| < L|x-y|\ \forall f \in \mathcal{E} \Leftarrow \exists M$ s.t. $|f'(x)| < M\ \forall f \in \mathcal{E}, \forall x$ Bounded Derivative

$\mathcal{E}$ uniformly equicontinuous if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall s, t \in [a,b]\ \forall f \in \mathcal{E}\ |s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$

[Arzela-Ascoli propagation thm]
$(f_n)$ converges pointwise on $\mathbb{Q} \cap [a,b]$
$(f_n)$ uniformly equicontinuous $\Rightarrow$
$f_n$ converges uniformly

$\mathcal{E}$ Compact $\Leftrightarrow \mathcal{E}$ closed, bounded and equicontinuous [Heine Borel Thm of $C_0$]

[Arzela Ascoli propagation thm] (closure of $(f_n)$ is compact)

$(f_n)$ equicontinuous and uniformly bounded $\Rightarrow (f_n)$ has uniformly convergent subsequence [Arzela-Ascoli Thm]

$\Leftrightarrow \exists p$ s.t. $(f_n(p))$ bounded in $\mathbb{R}$.

[A is any compact metric space]

$\mathcal{E}$ Dense in $C_0(A)$ if $\overline{\mathcal{E}} = C_0(A)$, $\overline{\mathcal{E}} = \{f \in C_0(A) \mid \forall \varepsilon > 0\ \exists g \in \mathcal{E} \text{ s.t. } \|f - g\| < \varepsilon\} = \{f \in C_0(A) \mid \exists (f_n) \subseteq \mathcal{E} \text{ s.t. } f_n \rightrightarrows f\}$

$\to \overline{\mathcal{E}}$ also function algebra

$\to f, g \in \overline{\mathcal{E}} \Rightarrow |f| \in \overline{\mathcal{E}}$, $\max(f,g) \in \overline{\mathcal{E}}$, $\min(f,g) \in \overline{\mathcal{E}}$

$\mathcal{E}$ Function Algebra if ① $\forall f, g \in \mathcal{E}$, $\forall c \in \mathbb{R}$, $f+g, fg, cf \in \mathcal{E}$ ② $\mathcal{E} \neq \emptyset$

$\mathcal{E}$ Vanishes Nowhere if $\forall p \in A$, $\exists f \in \mathcal{E}$ s.t. $f(p) \neq 0$

$\mathcal{E}$ Seperates points if $\forall p_1, p_2 \in A$, $\exists f \in \mathcal{E}$ s.t. $f(p_1) \neq f(p_2)$

$\Rightarrow \forall p_1 \neq p_2 \in A$, $\forall c_1, c_2 \in \mathbb{R}\ \exists f \in \mathcal{E}$ s.t. $f(p_1) = c_1$, $f(p_2) = c_2$

$\mathcal{E}$ function algebra, seperates points and vanishes nowhere $\Rightarrow \overline{\mathcal{E}} = C_0(A)$ [Stone-Weierstrass thm]

$T: M \to M$

weak contraction $d(f_x, f_y) < d(x,y)$

$\Uparrow$ Contraction if $\exists k < 1$ s.t. $\forall x, y\ d(Tx, Ty) < k\,d(x,y) \Rightarrow T$ uniformly continuous (contraction $\Rightarrow$ Lipschitz)

**Contraction Mapping Principal** $M$ compact, $T$ weak contraction $\Rightarrow$; $M$ complete, $T$ contraction $\Rightarrow T$ has unique fixed point ($x \in M$ s.t. $T(x) = x$).

*for continuity, $\delta$ depends on $\varepsilon, x_0, f \Leftarrow$ pointwise equicontinuity, $\delta$ depends on $\varepsilon$ and $x_0$

$\Uparrow$ uniform cont. $\delta$ depends on $\varepsilon, f \Leftarrow$ uniform equicontinuity, $\delta$ depends on $\varepsilon$ $\Uparrow$

$T: X \to Y$ bijection, $\nu(T(E)) = t\mu(E)\ \forall E \subset X$ for some $t > 0$.
$\mu(T^{-1}(B)) = \frac{1}{t}\nu(B)\ \forall B \subset Y$

Affine maps: $T: \mathbb{R}^n \to \mathbb{R}^n$, $Tx = Mx + x_0$ ($x_0$ translation, $M$ matrix)
$\hookrightarrow \nu(T(A)) = |\det T|\,\mu(A)$

$(X, F, \mu), (Y, S, \nu)$ Outermeasure spaces, $T: X \to Y$ Meseomorphism if $\forall A \in F$, $T(A) \in S$, $T$ Meseometry if $\nu(T(A)) = \mu(A)$

$M$ orthogonal $\Rightarrow T$ is a rigid motion $\Rightarrow \det T = 1$ (no stretching, only rotating)

**Outer Measures**

$m^*: 2^X \to [0, \infty]$ s.t. ① $m^*(\emptyset) = 0$ ② $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$ [Monotonicity] ③ $m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*(A_k)$ [Countably Subadditive]

example: **Lebesgue Outer Measure**

$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} |I_k| \;\middle|\; \bigcup_{k=1}^{\infty}\{I_k\} \text{ covering of } A \text{ by open intervals} \right\}$, $m^*: 2^M \to [0, \infty]$

Is an outer measure!, $m^*([a,b]) = b - a$

$(M, \Sigma, m^*|_{\Sigma})$ → $\Sigma = \{E \subseteq M \mid E$ measurable w.r.t. $m^*\}$ → $\forall X \subseteq M$, $m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$

**σ-algebra** $\Sigma \subseteq 2^M$ ① $\emptyset \in M$ ② $A \in \Sigma \Rightarrow A^c \in \Sigma$ ③ $\Sigma$ closed under countable unions.

**Measure Space**

$(M, \Sigma, \mu)$ **Measure** $\mu: \Sigma \to [0, \infty]$ ① $\mu(\emptyset) = 0$ ② $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ [Monotonicity] ③ $\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ [Countably additive]

$E_n \uparrow E$, $E_n \in \Sigma\ \forall n \Rightarrow \mu(E) = \lim_{n \to \infty} \mu(E_n)$

$B_n \downarrow B$, $B_n \in \Sigma\ \forall n$ $\mu(B_1) < \infty \Rightarrow \mu(B) = \lim_{n \to \infty} (B_n)$ [Notice $E_n \uparrow E$, $E = \bigcup_n E_n = \bigsqcup_k E'_k$, $E'_k = E_k \setminus \left(\bigcup_{i=1}^{k-1} E_i\right)$]

**Lebesgue Measure**

$(\mathbb{R}^n, \Sigma, \mu)$,
- $\mu$ translation invariant
- all open sets are in $\Sigma$
- $\mu$ complete [if $A \subseteq B \subseteq \mathbb{R}^n$, $\mu(B) = 0 \Rightarrow A \in \Sigma$]
- $\mu(\text{unit box}) = 1$

$\Rightarrow (\mathbb{R}^n, \Sigma, \mu)$ is Lebesgue measure

open in $\mathbb{R} \Rightarrow F_\sigma$, open $\Rightarrow G_\delta$; $G_\delta, F_\sigma$ closed under intersections and unions.
Closed in $\mathbb{R} \Rightarrow G_\delta$, Closed $\Rightarrow F_\sigma$

[regularity]
$E$ Lebesgue measurable $\Leftrightarrow \exists F_\sigma, G_\delta$ sets s.t. $F \subseteq E \subseteq G$, $G \setminus F$ is a Zero-set. $\Leftrightarrow E = F \cup Z$, $F$ a $F_\sigma$ set, $Z$ a zero-set
$\Leftrightarrow E = G \setminus Z$, $G$ a $G_\delta$ set, $Z$ a zero-set

$G_\delta$ → Countable intersection of open sets.
$F_\sigma$ → Countable union of closed sets.

measurable product thm: $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^k$ measurable, then so is $A \times B \subseteq \mathbb{R}^{n+k}$ and $m_{n+k}(A \times B) = m_n(A)\, m_k(B)$

Zero-slice thm: $E \subseteq \mathbb{R}^{n+k}$, $m_{n+k}(E) = 0 \Leftrightarrow \{x \in \mathbb{R}^n \mid m_k(E_x) \neq 0\}$ is a zero-set on $\mathbb{R}^n$. $E_x = \{y \in \mathbb{R}^k \mid (x, y) \in E\} \subseteq \mathbb{R}^k$

**Lebesgue Integral**

$f: \mathbb{R} \to [0, \infty)$, $f$ lebesgue measurable if $\mathcal{U}f = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y < f(x)\}$ is Lebesgue measurable $\Rightarrow \int f = m(\mathcal{U}f)$ ← Lebesgue Integral (Could be $\infty$)

$f$ lebesgue Integrable if $m_2(\mathcal{U}f) < \infty \Leftrightarrow \int f < \infty$.

$f = f_+ - f_-$ lebesgue measurable $\Leftrightarrow f_+$ and $f_-$ are, lebesgue integrable $\Leftrightarrow f_+$ and $f_-$ are.

$f, g$ Lebesgue measurable, then ① $f \leq g \Rightarrow \int f \leq \int g$ ② $\int cf = c\int f$ ③ $\int f + g = \int f + \int g$ ④ $A \subseteq \mathbb{R}$ measurable, $\int \chi_A = m(A)$
⑤ $A = \bigsqcup A_n \Rightarrow \int_A f = \sum_n \int_{A_n} f$ ⑥ $m(Z) = 0 \Rightarrow \int_Z f = 0$ ⑦ $f = g$ a.e. $\Rightarrow \int f = \int g$ ⑧ $\int f = 0 \Leftrightarrow f = 0$ a.e. (← Zero-slice thm.)

MCT: let $(f_n)$ seq. of +ve measurable functions, $f_n \uparrow f \Rightarrow f$ measurable, $\int f_n \to \int f$ $\left[\Rightarrow \int \sum^{\infty} f_n = \sum^{\infty} \int f_n\right]$

MCT2: $f_n \downarrow f$, $f_n$ integrable $\Rightarrow \int f_n \to \int f$, uses $\hat{\mathcal{U}}f = \{(x,y) \in \mathbb{R}^2 : 0 \leq f(x) \leq y\}$ and that $\mathcal{U}f$ measurable $\Leftrightarrow \hat{\mathcal{U}}f$ Measurable and $m_2\mathcal{U}f = m_2\hat{\mathcal{U}}f$.

Fatou Lemma: $\int \liminf f_n \leq \liminf \int f_n$

Dominated Convergence Thm: $f_n \to f$ a.e. $\exists g$ integrable and $f_n \leq g\ \forall n$ ($|f_n| \leq g$ if $f$ not positive!) $\Rightarrow f_n$ integrable $\forall n$, $f$ integrable and $\int f_n \to \int f$.

Cavalieri's Principle: Let $E \subseteq \mathbb{R}^{n+k}$ measurable,
① for almost all $x \in \mathbb{R}^n$, $E_x \subset \mathbb{R}^k$ measurable
② function $\mathbb{R}^n \to [0, \infty]$ $x \mapsto m_k(E_x)$ measurable
③ $m_{n+k}(E) = \int_{\mathbb{R}^n} m_k(E_x)\,dx$

Fubini-Tonelli: Let $f: \mathbb{R}^2 \to [0, \infty)$ measurable or $f: \mathbb{R}^2 \to \mathbb{R}$ integrable
① for almost all $x \in \mathbb{R}$, $f_x$ measurable
② function $\mathbb{R} \to \mathbb{R}$ $x \mapsto \int_{\mathbb{R}} f_x(y)\,dy$ measurable
③ $\iint_{\mathbb{R}^2} f = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_x(y)\,dy\right]dx$